Notes for p-adic representations

1 Motivation

Up until this point we have defined, for every reductive group $G/\mathbb{Q}$, two (highly infinite-dimensional) spaces of functions $G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ denoted $\mathscr{A}(G)$ and $\mathscr{A}_0(G)$ the space of *automorphic forms* for G and the space of *cuspidal autmorphic forms* for G respectively. Eventually we will (roughly) define an *automorphic representation* of G to be an irreducible subquotient of $G(\mathbb{A}_\mathbb{Q})$'s action on $\mathscr{A}(G)$ and a *cuspidal automorphic representation* of G to be an irreducible subquotient of $G(\mathbb{A}_\mathbb{Q})$'s action on $\mathscr{A}_0(G)$.

Remark 1.1: This is, technically, a lie. Namely there is a natural action of $K_\infty \times G(\mathbb{A}^\infty)$ on $\mathscr{A}(G)$ (where, here, K_∞ is a maximal compact subgroup of $G(\mathbb{R})$) but, unfortunately, this cannot be promoted to an action of $G(\mathbb{A}_\mathbb{Q})$ on $\mathscr{A}(G)$—right translation by $G(\mathbb{R})$ on functions $f : G(\mathbb{Q})/G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ does not preserve the subspace $\mathscr{A}(G)$. Thus, we will have to find a replacement (spoiler: instead of an honest-to-god $G(\mathbb{A}_\mathbb{Q})$ action we will get what's called a $(\mathfrak{g}_\mathbb{C}, K_\infty) \times G(\mathbb{A}^\infty)$-action). ♦

Now, the reason we are interested in doing this (or, rather, the reason *I* am interested in doing this) is because the relationship between automorphic representations and Galois representations posited by the Global Langlands Correspondence. Now, while we will not here state what this is precisely the idea is roughly the following:

Idea 1.2 (Global Langlands Correspondence for GL_2): There should be a correspondence:

$$\left\{\begin{matrix}\text{Cuspidal automorphic}\\ \text{representations for } \mathrm{GL}_2\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Irreducible representations}\\ G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})\end{matrix}\right\} \tag{1}$$

♦

Now, one of the key features of the right-hand side of this correspondence is the notion of *local representations*. Namely, every Galois representation $\rho : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$ comes equipped with a natural (conjugacy class) of local representations $\rho_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$ coming from the (conjugacy class) of embeddings $G_{\mathbb{Q}_p} \hookrightarrow G_\mathbb{Q}$.

So, if something like the Global Langlands Correspondence for GL_2 is to hold then there should be, on the left-hand side, an analogous 'local theory'. But, what precisely should this mean? Well, the rough idea is as follows:

Idea 1.3: It's well-known that $\mathrm{Irr}(G \times H) = \mathrm{Irr}(G) \otimes \mathrm{Irr}(H)$ (at least formally—say if G and H are finite groups and Rep means finite-dimensional representations). Now, we know that

$$G(\mathbb{A}_\mathbb{Q}) = G(\mathbb{R}) \times G(\mathbb{A}^\infty) = G(\mathbb{R}) \times \prod_p{}' G(\mathbb{Q}_p) \tag{2}$$

where, here, the prime in the product denotes the restricted product (with respect to $\mathbf{G}(\mathbb{Z}_p)$ where $\mathbf{G}$ is a model over G over $\mathrm{Spec}(\mathbb{Z}) - S$ for some finite set S of primes). Thus, formally, one might expect that if π is an automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ then we have a decomposition

$$\pi = \pi_\infty \otimes \bigotimes_p{}' \pi_p \tag{3}$$

(as in equation (2)) where π_∞ is a $G(\mathbb{R})$-representation (again, really, a $(\mathfrak{g}_\mathbb{C}, K_\infty)$-module) and π_p is a $G(\mathbb{Q}_p)$-representation (this is actually true (when interpreted correctly) by Flath's decomposition theorem). ♦

Thus, it seems that before we tackle the big, scary world that is $G(\mathbb{A}_\mathbb{Q})$-representations it behooves us to first understand these local representations π_∞ and π_p. As the title of this note might tip one off, we focus here on the π_p cases.

2 Basic definitions

We first start by defining what type of representations of $G(\mathbb{Q}_p)$ we are interested in. This is slightly less trivial than one might imagine. Namely, the groups $G(\mathbb{Q}_p)$ are extremely strange topologically thus one might imagine that what a 'continuous representation' of $G(\mathbb{Q}_p)$ on a complex vector space is will be somewhat strange. Also, note that, in general, our representations are going to *infinite dimensional* (recall that, in a holistic/rough sense, we're really trying to decompose the wildly infinite-dimensional space $L^2(G(\mathbb{Q})/G(\mathbb{A}_\mathbb{Q}))$—or, more rigorously, the K-finite vectors in this space are a direct integral of the spaces of interest to us) so things are bound to get even more strange.

So, before we actually define the class of representations that will interest us, let us define a class of groups which comfortably contains the groups $G(\mathbb{Q}_p)$, acts the way 'we want' and, more importantly, is closed under taking closed subgroups (so it will contain things like $\mathbf{G}(\mathbb{Z}_p)$):

Definition 2.1: Let G be a topological group. We call G *TD* (for 'totally disconnected') if

1. G has a neighborhood basis of the identity consisting of profinite subgroups.
2. For one (equivalently for all) compact open subgroups $K \subseteq G$ the index $[G : K]$ is countable. ▮

Perhaps a better phrase than 'TD' (which was lifted from notes of Brian Conrad) would be 'locally profinite'. Also, condition 2. is just for purely technical convenience since it makes things like Schur's lemma work—one should just ignore it in practice.

Of course we have the following:

Example 2.2: Let $G/\mathbb{Q}_p$ be a linear algebraic group. Then, $G(\mathbb{Q}_p)$ is a TD group. Moreover, for any model $\mathbf{G}/\mathbb{Z}_p$ the group $\mathbf{G}(\mathbb{Z}_p)$ is TD. ▮

One should note the obvious (and alluded to) fact that a closed subgroup of a TD group is TD.

So, now that we know the type of groups that we are interested in, let's define the types of representations of these groups that matter to us:

Definition 2.3: Let V be a complex vector space possibly (...probably) of infinite-dimension. A linear action of G on V (i.e. a representation—a homomorphism $\rho : G \to \mathrm{GL}(V)$) is called *smooth* if for all $v \in V$ the subgroup $\mathrm{stab}(v) \subseteq V$ is open. If, in addition, V^K is finite-dimensional for all compact open subgroups $K \subseteq V$ then we call V (or ρ) *admissible.* ▮

One nice way of thinking about both of these conditions simultaneously is to note that if $\rho : G \to \mathrm{GL}(V)$ is smooth then

$$V = \bigcup_K V^K \tag{4}$$

as K ranges over the compact open subgroups of G. If, in addition, each V^K is finite-dimensional then ρ (or V) is admissible.

So, with this being said we have the following:

Goal: *Understand the irreducible admissible representations of* $G(\mathbb{Q}_p)$*...really just* $G = \mathrm{GL}_2$ *in these notes.*

Why are *these* the representations that we single out? well, intuitively smoothness is the replacement for 'continuous homomorphism' in the context of infinite-dimensional $\mathbb{C}$-spaces which have no canonical topology (of course, one can think about unitary representations of G on Hilbert spaces, and we do, but that is a topic for another talk). For example, if V is finite-dimensional (so does have a canonical topology) then $\rho : G \to \mathrm{GL}(V)$ is smooth if and only if ρ is continuous. Admissibility is a finiteness condition which says that V is somehow 'built' out of finite-dimensional irreducible representations of K (for any compact open K) which allows one to deduce a lot.

To this effect we have the following alternative characterization of admissibility:

Theorem 2.5: *Let $G/\mathbb{Q}_p$ be reductive and $\mathbf{G}/\mathbb{Z}_p$ a reductive model and let $K \subseteq G(\mathbb{Q}_p)$ be equal to $\mathbf{G}(\mathbb{Z}_p)$ (a so-called hyperspecial subgroup of $G(\mathbb{Q}_p)$). Then, for any $\rho : G \to \mathrm{GL}(V)$ a representation the following are equivalent:*

1. *ρ is admissible.*
2. *$V = \bigoplus_{\sigma \in \mathrm{Irr}(K)} V[\sigma]$ (where $V[\sigma]$ is the σ-isotypic component of V) and $V[\sigma]$ is finite-dimensional.*

So, for our prime example of $G = \mathrm{GL}_2(\mathbb{Q}_p)$, so that $K = \mathrm{GL}_2(\mathbb{Z}_p)$, the above says that admissible representations are essentially direct sums of (possibly infinitely many) K-representations, but each isotypic component for K is finite-dimensional. This makes dealing with 'character theory' on V possible since if V is admissible then it makes sense as a 'virtual power series' over K (i.e. an infinite sum of characters of K)—this would not possible if $V[\sigma]$ were infinite-dimensional since the coefficients of this 'virtual character power series' would no longer be integers (more precisely in the parlance of Hecke algebras that we'll discuss next time: admissible representations are those for which the Hecke action is by finite-rank operators)

Remark 2.6: It's probably true that in Theorem 2.5 one can take G to be TD and K to be any maximal compact open, but I haven't checked this. ♦

The other reason that we care about admissible representations of TD groups is that, well, the local factors π_p, as representations of $G(\mathbb{Q}_p)$, for an automorphic representation π of $G(\mathbb{A}_\mathbb{Q})$ are admissible!

Now we will not need too much general theory about admissible representations of TD groups, but we mention here three important results:

Theorem 2.7 (Schur's lemma): *Let V be an irreducible admissible representation of the TD group G. Then, $\mathrm{End}(V) = \mathbb{C}$.*

This, incidentally, is where we need the second condition (on the indices of compact open subgroups). As a natural corollary of this we see that if V is an irreducible admissible representation of G and $Z := Z(G)$ then we get a natural central character $\omega = \omega_V : Z \to \mathbb{C}^\times$.

Theorem 2.8: *Let V be an irreducible admissible representation of a TD group G. Then, $\dim V$ is countable.*

Thus, irreducible admissible representations of TD groups can't get 'too big'—this is also a consequence of the second condition in TD groups.

Theorem 2.9: *Let $G/\mathbb{Q}_p$ be a reductive group. Then, every irreducible smooth representation of $G(\mathbb{Q}_p)$ is admissible.*

This is a somewhat deep (although the proof is not too hard) theorem saying that admissbility is sort of a non-issue for our current study.

3 Finite-dimensional representations

So, the first thing we might do is try and classify the irreducible finite-dimensional smooth (=admissible for obvious reasons) representations of G a TD group. The key result is the following:

Theorem 3.1: *Let $G/\mathbb{Q}_p$ be a $\mathbb{Q}_p$-split reductive group ($\mathbb{Q}_p$-split just means that it has a maximal torus [recall that maximal here means maximal over $\overline{\mathbb{Q}_p}$] which is split over $\mathbb{Q}_p$). Then every finite-dimensional smooth irreducible representation of $G(\mathbb{Q}_p)$ is 1-dimensional.*

So, for example, this applies in the case of $G = \mathrm{GL}_2$ so that every finite-dimensional irreducible representation is a character. That said, in the case of $G = \mathrm{GL}_2$ there is a much more tame argument. Essentially, if $\rho : G \to \mathrm{GL}(V)$ is such a representation then $\ker \rho$ is a compact open normal subgroup of G. But, one can check that this implies that $\ker\rho$ is $\mathrm{SL}_2(\mathbb{Q}_p)$. The idea: show that the normal subgroup generated by $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$ for any x is $\mathrm{SL}_2(\mathbb{Q}_p)$ then note that $\ker\rho$, being open, must contains $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$ for x sufficiently small. In fact, one can show (using the structure theory of k-split G) that the GL_2 case implies the general case!

Thus, we see that for essentially all the TD groups we care about the finite-dimensional irreducible admissible representations are just characters. But, one must be careful! Namely, it's tempting to say that in the setting of Theorem 3.1 any finite-dimensional representation $\rho : G(\mathbb{Q}_p) \to \mathbb{C}^\times$ must factor through $(G/D(G))(\mathbb{Q}_p)$ (where $D(G)$ is the derived subgroup of G) which is great since $G/D(G)$ is a torus. That said this is not true in general. Namely, recall that the equality $D(G)(K) = D(G(K))$ holds (necessarily) only when K is separably closed (for example when $G = \mathrm{GL}_2$ and $K = \mathbb{F}_2$ we have that $D(G) = \mathrm{SL}_2$ but $D(\mathrm{GL}_2(\mathbb{F}_2)) \neq \mathrm{SL}_2(\mathbb{F}_2)$). So, we cannot say that all characters of $G(\mathbb{Q}_p)$ factors through the $\mathbb{Q}_p$-points of a torus (the torus $G/D(G)$).

That said, when $G = \mathrm{GL}_n$ we *can* make this claim. Indeed, it *is* true that $D(\mathrm{GL}_n(K)) = \mathrm{SL}_n(K)$ when, for example, K is infinite which is the case we are in. Thus, we can use Theorem 3.1 to give a complete characterization of finite-dimensional irreducible admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$:

Example 3.2: Let us now classify all the finite-dimensional irreducible admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ or, equivalently by Theorem 3.1, the smooth(=continuous) characters $\chi : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}^\times$. Note that we have the following equality:

$$D(\mathrm{GL}_n(\mathbb{Q}_p)) = \mathrm{SL}_n(\mathbb{Q}_p)$$

so that every continuous character $\chi : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}^\times$ factors as

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{Q}_p) & \xrightarrow{\chi} & \mathbb{C}^\times \\ & \searrow^{\det} & \uparrow \overline{\chi} \\ & & \mathbb{Q}_p^\times \end{array}$$

thus, really, we only need to classify the smooth characters χ of $\mathbb{Q}_p^\times$ for then every character of $\mathrm{GL}_n(\mathbb{Q}_p)$ is of the form $g \mapsto \chi(\det(g))$ for some such χ.

Now, note that we have a natural decomposition (as topological groups) $\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times$. Thus, characters of $\mathbb{Q}_p^\times$ amounts to products of continuous characters of $\mathbb{Z}$ and continuous characters of $\mathbb{Z}_p^\times$. In other words, if we use a hat to denote the character group of a (locally compact abelian) group then we have the equality

$$\widehat{\mathbb{Q}_p^\times} \times \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}_p^\times}$$

Now, $\widehat{\mathbb{Z}}$ is simple—it's just $\mathbb{C}^\times$ (note that it's not S^1 since we're not dealing with the usual character group seen in harmonic analysis which is the *unitary* character group—here our characters are not required to be unitary) and thus we really only need to understand $\widehat{\mathbb{Z}_p^\times}$.

Now, let's consider a continuous character $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$. Then, by smoothness, we know that $\ker\chi \subseteq \mathbb{Z}_p^\times$ is open and thus contains $1 + p^n\mathbb{Z}_p$ for some n. Let us call the smallest n such that $1 + p^n\mathbb{Z}_p \subseteq \ker\chi$ the *conductor* of χ (conductor 0 should be interpreted to mean that it's trivial on $\mathbb{Z}_p^\times$). Then, we see that if χ has conductor n then we have a factorization

$$\begin{array}{ccc} \mathbb{Z}_p^\times & \xrightarrow{\chi} & \mathbb{C}^\times \\ & \searrow & \uparrow \overline{\chi} \\ & & (\mathbb{Z}/p^n\mathbb{Z})^\times \end{array}$$

thus we're really down to understanding characters of $(\mathbb{Z}/p^n\mathbb{Z})^\times$. That said, the structure of this group is simple. Namely, if $p \neq 2$ then

$$(\mathbb{Z}/p^n\mathbb{Z})^\times \cong \mathbb{Z}/p^{n-1}(p-1)\mathbb{Z}$$

and if $p = 2$ then

$$(\mathbb{Z}/p^n\mathbb{Z})^\times \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2^{n-2}\mathbb{Z})$$

and let us fix such isomorphisms.

So, we can summarize all of this as above. Let $p \neq 2$ be a prime. Then, giving a character of $\mathrm{GL}_n(\mathbb{Q}_p)$ amounts to giving a triple (s, m, ζ) where $s \in \mathbb{C}^\times$, $m \in \mathbb{N} \cup \{0\}$, and $\zeta \in \mu_{p^{m-1}(p-1)}(\mathbb{C})$. In particular if $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ and if $\det(g) = p^k u$ (with $u \in \mathbb{Z}_p^\times$) then $\chi_{(s,m,\zeta)}(g) = s^k \zeta^\ell$ if $u \mod p^m = c^\ell$ where c is our chosen generator of $(\mathbb{Z}/p^m\mathbb{Z})^\times$. A similar statement can make when $p = 2$ except we really see that characters are equivalent to quadruples $(s, m, \zeta, \pm 1)$ with the obvious definitions.

4 A first real example

Now that we have essentially sussed out the finite-dimensional irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ let us give what, at this point, probably seems like a hopelessly scary *infinite-dimensional* example. It will turn out that this example is important for theoretical reasons that we will emphasize later.

So, to intuit how one might go about creating such an example, let us note that a natural way of getting a $\mathrm{GL}_2(\mathbb{Q}_p)$-action on a $\mathbb{C}$-space is to first find a space X on which $\mathrm{GL}_2(\mathbb{Q}_p)$ acts and then think about $\mathrm{GL}_2(\mathbb{Q}_p)$ acting on the space $\mathrm{Fun}(X, \mathbb{C})$ of $\mathbb{C}$-valued functions on X or, perhaps less perversely, on the space $C(X)$ of continuous $\mathbb{C}$-valued functions on X. So, how can we go about creating a space X on which $\mathrm{GL}_2(\mathbb{Q}_p)$-acts (acts on the right)? Well, if we hope for the space $C(X)$ to be an irreducible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation then we better have that $\mathrm{GL}_2(\mathbb{Q}_p)$'s action on X is 'large' (if we can't move between most points of X by $\mathrm{GL}_2(\mathbb{Q}_p)$ how can we hope to move between most functions in $C(X)$ by a $\mathrm{GL}_2(\mathbb{Q}_p)$-action?). In particular, we probably want that $\mathrm{GL}_2(\mathbb{Q}_p)$ acts transitively on X so that it becomes a homogenous space $X \cong H\backslash \mathrm{GL}_2(\mathbb{Q}_p)$ (we want to quotient on the left since our group is going to be acting on the right) for $H = \mathrm{stab}(x)$ (for any $x \in X$).

So we have reduced ourselves from finding an X to finding an H. What kind of H will suffice? Well as is well-known the space $C(X)$ is a bit wild unless X is compact and thus, perhaps, we want to find H such that $H\backslash \mathrm{GL}_2(\mathbb{Q}_p)$ is compact. But, also, it'd be nice if H itself was 'algebraic' (since we understand those groups the best) in the sense that $H = P(\mathbb{Q}_p)$ for some algebraic subgroup $P \subseteq \mathrm{GL}_2$. Thus, we want some subgroup $P \subseteq \mathrm{GL}_2$ such that $P(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)$ is compact and thus, in good situations, $(P\backslash \mathrm{GL}_2)(\mathbb{Q}_p)$ is compact (NB: I'll mention this more below but, of course, in general, $(G\backslash H)(R) \neq G(R)\backslash H(R)$). This then strongly suggests that we'd want $P\backslash \mathrm{GL}_2$ to be proper or, since it's automatically quasi-projective, projective. Thus, all-in-all we see that are looking for $P \subseteq \mathrm{GL}_2$ where $P\backslash \mathrm{GL}_2$ is projective and, as it were, these have a name—they are the *parabolic* subgroups of GL_2.

Now, as it turns out, there is essentially 'only one' parabolic subgroup of GL_2 (really a unique one up to conjugation) which is the subgroup

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_2$$

of upper-triangular matrices—the *standard Borel.* And, moreover, in this case we have that $B\backslash \mathrm{GL}_2 \cong \mathbb{P}^1$ and so $(B\backslash G)(\mathbb{Q}_p) = \mathbb{P}^1(\mathbb{Q}_p) = \mathbb{Q}_p \cup \{\infty\}$ (i.e. the one-point compactification of $\mathbb{Q}_p$) and since $(B\backslash G)(\mathbb{Q}_p) = B(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$ (in this case) we see that a natural place we might look for an interesting representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is in $C(\mathbb{P}^1(\mathbb{Q}_p))$.

Remark 4.1: As mentioned above one must be careful with questions about whether $(G/H)(R) = G(R)/H(R)$ (where I'm going to slip into right quotients for a second) for G an algebraic group (say over $\mathbb{Q}_p$), H an algebraic subgroup, and R a $\mathbb{Q}_p$-algebra. In general one only has an exact sequence (noting that H is automatically smooth so that fppf cohomology with H-coefficients agrees with étale cohomology):

$$0 \to H(R) \to G(R) \to (G/H)(R) \to H^1_{\acute{e}t}(\mathrm{Spec}(R), H)$$

so, for example, in our case this was a non-issue since $H^1_{\acute{e}t}(\mathrm{Spec}(\mathbb{Q}_p), B) = 0$. Indeed, note that $B(\overline{\mathbb{Q}_p})$ fits into an exact sequence

$$1 \to \left(\overline{\mathbb{Q}_p}^\times\right)^2 \to B(\overline{\mathbb{Q}_p}) \to \overline{\mathbb{Q}_p} \to 1$$

(given by considering diagonal matrices and their quotient) and thus we have that the Galois cohomology group $H^1_{\acute{e}t}(\mathrm{Spec}(\mathbb{Q}_p), B)$ is trivial. Of course, this is all overkill here since one can just show directly that $B(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p) = \mathbb{P}^1(\mathbb{Q}_p)$ but it's good to see how this example sits in the larger theory. ♦

So, let us note that we can rewrite $C(\mathbb{P}^1(\mathbb{Q}_p))$ in slightly nicer notation. Namely, since $\mathbb{Q}_p$ has such incommensurate topology with $\mathbb{C}$ one can check that continuous functions $f : \mathbb{P}^1(\mathbb{Q}_p) \to \mathbb{C}$ are automatically locally constant or, in the parlance of p-adic analysis, smooth. Moreover, since $\mathbb{P}^1(\mathbb{Q}_p)$ is compact the support of any f is necessarily compact. Thus, $C(\mathbb{P}^1(\mathbb{Q}_p))$ might also be denoted $C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ (i.e. smooth, compactly supported functions $\mathbb{P}^1(\mathbb{Q}_p) \to \mathbb{C}$). This observation is purely cosmetic but it's helpful to match to the usual notation used for this representation.

It's fairly trivial (using the local constantness of our functions and their compact support) that $C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ is an admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Now, a naive hope is that it's irreducible but, alas, this is patently false. Specifically, $\mathrm{GL}_2(\mathbb{Q}_p)$ certainly stabilizes the 1-dimensional subspace of $C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ consisting of constant functions. Somewhat surprisingly, this is the 'only impediment' to irreducibility:

Theorem 4.2: *The representation $C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))/\rho_{\mathrm{triv}}$ (where the trivial representation is embedded as the constant functions) is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.*

The proof is fairly simple and comes down to the observation that $\mathrm{GL}_2(\mathbb{Q}_p)$ essentially acts transitively on the compact subsets of $\mathbb{P}^1(\mathbb{Q}_p)$. Noting that $\dim C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ is $\aleph_0$ we see that we have now produced an example of an infinite-dimensional irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

5 Induction and compact induction

Now that we have seen *some* examples of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ then next natural question is whether or not we can manufacture a general technique to create them—a method that will allow us to get a whole *host* of examples. Of course, the first place one looks in such a situation is to subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$. Namely, a well-known technique in the study of representations of groups is the use of *induction* which allows us to take a representation of some subgroup $H \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$ and 'lift' it to a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. More care must be taken here though since, of course, if $[\mathrm{GL}_2(\mathbb{Q}_p) : H]$ is infinite then the induced representation can, *a priori*, get pretty wild and might break smoothness/admissibility properties of the original representation.

So, let us start with a bare bones definition:

Definition 5.1: Let G and H be any groups with $H \subseteq G$ and let $\rho : H \to \mathrm{GL}(V)$ be a representation. We define the *induced representation* of ρ, denoted $\mathrm{Ind}_H^G \rho$ or $\mathrm{Ind}_H^G V$, to be the space

$$\mathrm{Ind}_H^G V = \{f : G \to V : f(hg) = \rho(h)(f(g)) \text{ for all } h \in H,\, g \in G\}$$

for which G acts on by right translation. ▮

Now, as mentioned above, one must be careful since even if V is a smooth representation of H then there is no guarantee that $\mathrm{Ind}_H^G V$ is a smooth representation of G (in fact, this essentially *never* happens) so we must modify our definition when dealing with smooth representations of TD groups. Namely:

Definition 5.2: Let G be a TD group and H a closed subgroup. Let $\rho : H \to \mathrm{GL}(V)$ be a smooth representation. Define the *smooth induction* of V to G, denoted $\text{sm-Ind}_H^G(V)$, to be the set of smooth vectors in $\mathrm{Ind}_H^G V$—the set of elements of $\mathrm{Ind}_H^G V$ with open stabilizers. Equivalently $\text{sm-Ind}_H^G V$ is the set

$$\left\{ f : G \to V : \begin{array}{ll} (1) & f(hg) = \rho(h)(f(g)) \text{ for all } h \in H,\, g \in G \\ (2) & f \text{ is locally constant} \end{array} \right\}$$ ▮

Then we see that, by definition, $\text{sm-Ind}_H^G V$ is a smooth representation of G. That said, it can happen that $\text{sm-Ind}_H^G V$ can fail to be admissible even if V is. Thus, with an eye towards admissible representations we need to modify sm-Ind_H^G even more:

Definition 5.3: Let G be a TD group and $H \subseteq G$ a closed subgroup. Suppose that $\rho : H \to \mathrm{GL}(V)$ is an admissible representation of V. Then, define the *compact induction* of ρ or V, denoted $\text{c-Ind}_H^G \rho$ or $\text{c-Ind}_H^G V$, as follows:

$$\text{c-Ind}_H^G V := \left\{ f \in \text{sm-Ind}_H^G V : \text{The image of } \mathrm{supp}(f) \text{ in } G/H \text{ is compact} \right\}$$

▌

Let us then list the basic properties of these various inductions that show they are, in fact, reasonable:

Theorem 5.4: *Let G be a TD group, H a closed subgroup, and $\rho : H \to GL(V)$ a representation. Then:*

1. $\text{sm-Ind}_H^G V$ *is a smooth G-representation if V is a smooth H-representation.*
2. $\text{c-Ind}_H^G V$ *is an admissible G-representation if V is an admissible H-representation.*
3. $\text{sm-Ind}_H^G V = \text{c-Ind}_H^G V$ *if G/H is compact.*
4. *Frobenius reciprocity holds for smooth induction:*

$$\mathrm{Hom}_G(W, \text{sm-Ind}_H^G V) = \mathrm{Hom}_H(W, V)$$

for all smooth G-representations W.

5. *Frobenius reciprocity (with role reversal) holds for compact induction:*

$$\mathrm{Hom}_G(\text{c-Ind}_H^G V, W) = \mathrm{Hom}_H(G, W)$$

for all admissible G-representations W.

6 Principal series representations

Let us turn the crank of the machine we developed in the last section to produce some interesting representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. So, what subgroup should we start with? Well, it probably behooves us to start in the basic case when $G\backslash H$ is compact (so smooth and compact induction agree) and, again seeing that it would be nice if H itself is algebraic, we are back in the situation of Section 4. Namely, we should probably consider inducing admissible representations from $B(\mathbb{Q}_p) \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$.

That said, one should be careful. Namely, B is *not* a reductive group, and so doesn't play by the same rules as the types of groups we've been dealing with above. Namely, it has a unipotent part (it's unipotent radical $R_u(B)$ are upper triangular matrices with 1's along the diagonal) and this has terrible representation theory which propagates to B. Thus, to hope that we get 'nice' representations coming from $B(\mathbb{Q}_p)$ perhaps the reasonable thing to do is to sort of only look at representations of it which are trivial on its unipotent radical—in other words, we're interested in representations on $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right\}$ which 'ignore' b. In other words, we want to take representations of $\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right\}$ and extend them to $B(\mathbb{Q}_p)$ by having them act trivially on 'the rest' (this is doable since B has the *Levi decomposition* $B = DN$ where D are the diagonal matrices and $N = R_u(B)$).

That said, note that $D = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right\}$ is abelian, and so all of its irreducible admissible representations are just characters. Thus, what we really see the first natural place to find representations is to take a pair of characters $\chi_i : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ (for $i = 1, 2$) extend this to a character of D as $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$, extend this to B by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$ and then induce this up to $\mathrm{GL}_2(\mathbb{Q}_p)$.

So, to this end, let us make the following definition:

Definition 6.1: Let χ_i for $i = 1, 2$ be a pair of characters of $\mathbb{Q}_p^\times$. Let $\rho_{(\chi_1, \chi_2)}$ be the representation on $B(\mathbb{Q}_p)$ defined by

$$\rho_{(\chi_1,\chi_2)}\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\right) = \chi_1(a)\chi_2(d)\left|\frac{a}{b}\right|_p^{\frac{1}{2}}$$

and define the *principal series* representation associated to (χ_1, χ_2), denoted $P(\chi_1, \chi_2)$, to be $\text{sm-Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}\rho_{(\chi_1,\chi_2)}$. ▮

The strange normalization in the above is standard and is mostly there not out of necessity, but out of convenience—it makes some formulas that appear later nicer looking.

Now, we are are really after irreducible smooth representations of $\text{GL}_2(\mathbb{Q}_p)$ and while certainly $\rho_{(\chi_1,\chi_2)}$ is irreducible there is no reason whatsoever to believe that $P(\chi_1, \chi_2)$ is. That said, it turns out that a) it is 'quite often' and b) when it's not it's decomposition into irreducibles is quite simple:

Theorem 6.2: *Let (χ_1, χ_2) be a pair of characters of $\mathbb{Q}_p^\times$.*

1. *If $\chi_1\chi_2^{-1} \neq |\cdot|_p^{\pm 1}$ then $P(\chi_1, \chi_2)$ is irreducible.*
2. *If $\chi_1\chi_2^{-1} = |\cdot|_p$ then $P(\chi_1, \chi_2)$ has a unique irreducible submodule of codimension 1.*
3. *If $\chi_1\chi_2^{-1} = |\cdot|_p^{-1}$ then $P(\chi_1, \chi_2)$ has a unique irreducible quotient with 1-dimensional kernel.*

In all cases we denote the above mentioned irreducible associated with $P(\chi_1, \chi_2)$ by $\pi(\chi_1, \chi_2)$. If $P(\chi_1, \chi_2) \neq \pi(\chi_1, \chi_2)$ we call $\pi(\chi_1, \chi_2)$ *special*.

Let us now consider an example of this construction. Namely, let's consider $\pi(|\cdot|_p^{\frac{-1}{2}}, |\cdot|_p^{\frac{1}{2}})$. Since, in this case, $\chi_1\chi_2^{-1} = |\cdot|_p^{-1}$ we know that $P(\chi_1, \chi_2)$ will have a unique irreducible quotient with 1-dimensional kernel. But, note that

$$P(|\cdot|_p^{\frac{-1}{2}}, |\cdot|_p^{\frac{1}{2}}) = \text{sm-Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}\rho_{\text{triv}} = C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p))$$

where the last equality follows from staring at the definition of $\text{sm-Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}\rho_{\text{triv}}$ for a second. Thus, the unique irreducible quotient with 1-dimensional kernel is *precisely* the representation we constructed in section 4. We call this the *Steinberg* representation of GL_2 and denote it St_{GL_2}. Now that for any other special representation $\pi_1(\chi_1, \chi_2)$ one can write it uniquely as $\text{St}_{\text{GL}_2} \otimes \chi$ for a character χ of $\text{GL}_2(\mathbb{Q}_p)$.

7 Supercuspidals and the classification

So, up until this point we've given essentially three classes of representations of $\text{GL}_2(\mathbb{Q}_p)$: characters, irreducible principal series, and special representations (i.e. twists of the Steinberg representation). One might wonder what else there is? Well, that is the wild world of supercuspidal representations.

Before we define these rigorously let us give the idea:

Idea 7.1: Supercuspidal representations are those that don't 'come from' a smaller group than GL_2. They are, in some sense, truly native to GL_2. ♦

In other words, exactly representations like $\pi(\chi_1, \chi_2)$ are the opposite of supercuspidals—they come from the smaller subgroup $B(\mathbb{Q}_p)$ (or, rather, the smaller reductive group $D = \mathbb{G}_{m,\mathbb{Q}_p}^2$) of $\text{GL}_2(\mathbb{Q}_p)$. One might wonder if they are, essentially, the only type of representations one has to avoid to be supercuspidal.

To this end:

Theorem 7.2: *Let $\rho : \text{GL}_2(\mathbb{Q}_p) \to \text{GL}(V)$ be an irreducible smooth representation. Then, the following are equivalent:*

1. *V is not isomorphic to a subquotient of $P(\chi_1, \chi_2)$ for a pair (χ_1, χ_2) of characters of $\mathbb{Q}_p^\times$.*
2. *V is not isomorphic to a subquotient of a representation induced from any parabolic subgroup of $\text{GL}_2(\mathbb{Q}_p)$.*

Note as an example of this that all characters of $\text{GL}_2(\mathbb{Q}_p)$ actually show up as subrepresentations of some principal series. Namely, since $P(|\cdot|_p^{\frac{-1}{2}}, |\cdot|_p^{\frac{1}{2}})$ contains a copy of the trivial representation and we have that

$$P(\chi|\cdot|_p^{\frac{-1}{2}}, \chi|\cdot|_p^{\frac{1}{2}}) = (\det\circ\chi)\otimes P(|\cdot|_p^{\frac{-1}{2}}, |\cdot|_p^{\frac{1}{2}})$$

we conclude that $P(\chi|\cdot|_p^{\frac{-1}{2}}, \chi|\cdot|_p^{\frac{1}{2}})$ contains a copy of χ.

Let us then, given this theorem, make the following definition:

Definition 7.3: Let $\rho : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{GL}(V)$ be an irreducible smooth representation. Call ρ *supercuspidal* if any of the equivalent conditions in Theorem 7.2 hold. ▌

Remark 7.4: There are much more elegant, useful definitions of supercuspidals involving the highly-powerful machine of the Jacquet functor, but given the time we have we have opted for this approach. ⧫

The importance for supercuspidals is immense. Indeed, under the Local Langlands Correspondence (the local analogue of Idea 1.2) they correspond to the *irreducible* Galois representations. Indeed, if a Galois representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$ is not irreducible then (up to semisimplification) it comes from a pair of characters and so, one would imagine, through the looking glass of the Local Langlands Correspondence it's associated irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ would come from a pair of characters—it would be *not* supercuspidal (in fact, it would probably come from a principal series representation!).

Given our somewhat anticlimactic definition the following theorem (which is usually somewhat shocking given the more standard definition of supercuspidals as having compact support mod center matrix coefficients) is not at all surprising:

Theorem 7.5 (Classification of irreducible admissible representations of $\mathbf{GL_2}(\mathbb{Q}_p)$): *Let $\rho : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{GL}(V)$ be a smooth irreducible representation. Then, ρ is one of the following:*

1. *A character of $\mathrm{GL}_2(\mathbb{Q}_p)$.*
2. *An irreducible principal series $P(\chi_1, \chi_2)$.*
3. *A special representation $\mathrm{St}_{\mathrm{GL}_2} \otimes \chi$ for χ a character of $\mathrm{GL}_2(\mathbb{Q}_p)$.*
4. *A supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.*

Moreover, this list is irredundant except $P(\chi_1, \chi_2) \cong P(\chi_2, \chi_1)$.

So, of course, the next question is: what *is* an example of a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$? Well, perhaps not shockingly, they are fairly hard to write down. Thinking Langlands-y this should correspond to writing down an irreducible 2-dimensional ℓ-adic representation of $G_{\mathbb{Q}_p}$ which (while certainly doable!) is somewhat hard. In a later talk we'll see that perhaps the easiest way to create such a supercuspidal representations (in a natural way) is to look at the p-adic local component of a modular form (one has to be choosey with the form though—many forms will not have supercuspidal p-adic component).

That said, we can write down a silly example just to wet one's whistle. Namely, let $\mathrm{sgn} : S_3 \to \{\pm 1\}$ be the sign character of S_3. Note that $S_3 = \mathrm{GL}_2(\mathbb{F}_2)$ and inflate sgn to a representation of $\mathrm{GL}_2(\mathbb{Z}_2)$ by defining $\rho(g) := \mathrm{sgn}(\overline{g})$ if $\overline{g}$ denotes g's image in $\mathrm{GL}_2(\mathbb{F}_2)$. Finally one can then show that $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_2)}^{\mathrm{GL}_2(\mathbb{Q}_2)}\rho$ is a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_2)$.

This is how many (the 'depth 0 supercuspidal') representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ occur. Take a 'supercuspidal' representation of $\mathrm{GL}_2(\mathbb{F}_p)$, inflate it in the obvious way to a representation of $\mathrm{GL}_2(\mathbb{Z}_p)$, and take the compact induction to get a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Here a 'supercuspidal' representation of $\mathrm{GL}_2(\mathbb{F}_p)$ means one where $\left\{\begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}\right\} \subseteq \mathrm{GL}_2(\mathbb{F}_p)$ has no fixed vector.

Thankfully, one can actually (using somewhat complicated combinatorics) actually enumerate *all* the supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, but this requires a huge amount of effort.

8 Unramified representations of and L-functions

Because we will need it for the next talk, we end this note by defining and describing the unramified representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. But, before we do so let us begin by recalling what it means for a character

$\chi : \mathbb{Q}_p^\times \to \mathbb{Z}_p^\times$ to be unramified. Namely, there it means precisely that $\mathbb{Z}_p^\times$ has a fixed vector or, equivalently, that $\mathbb{Z}_p^\times$ acts trivially on $\mathbb{C}$—in the parlance of Example 3.2 it's the characters of the form $(s, 0, 1)$. This makes sense since, under the Local Langlands Correspondence for $n = 1$ (i.e. local class field theory) the inertia subgroup of $G_{\mathbb{Q}_p}$ corresponds to $\mathbb{Z}_p^\times$ and thus unramified *should* mean that $\mathbb{Z}_p^\times$ acts trivially. We then extend the above definition to $\mathrm{GL}_2(\mathbb{Q}_p)$ by declaring that an admissible representation $\rho : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{GL}(V)$ to be *unramified* if $V^{\mathrm{GL}_2(\mathbb{Z}_p)} \neq 0$.

It turns out (and this is not too difficult to show) that all irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ come from the 1-dimensional case:

Theorem 8.1: *Let $\rho : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{GL}(V)$ be an irreducible admissible unramified representation. Then, $\rho = P(\chi_1, \chi_2)$ for χ_1 and χ_2 unramified characters of $\mathbb{Q}_p^\times$.*

This is, in some sense, not shocking. Or, rather, it should be totally expected that unramified representations are not supercuspidal. Why? Well they correspond, under the Local Langlands Correspondence, to irreducible Galois representations (again, roughly—I haven't stated this rigorously in terms of Weil-Deligne representations). And, it's not hard to see that an irreducible 2-dimensional $G_{\mathbb{Q}_p}$-representation cannot be unramified. Why one can't get a Steinberg is a little more complicated (it really depends on the more careful claim of the Local Langlands Correspondence involving Weil-Deligne representations).

As a corollary of Theorem 8.1 we see that an unramified representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is determined by a pair of complex numbers. Namely, if χ_1 is $(s, 0, 1)$ and $\chi_2 = (t, 0, 1)$ (in the parlance, again, of Example 3.2) then the pair of numbers is just (s, t). Again, this is totally expected—a 2-dimensional unramified $G_{\mathbb{Q}_p}$-representation is determined by a pair of numbers—the eigenvalues of Frobenius. Moreover, not shockingly, under the Local Langlands Correspondence these pairs of numbers 'match up'.

Finally, we end this note by giving an ad hoc definition of the L-function of *most* of the irreducible admissible representations we've defined. The reason it's ad hoc is (just like the definition of supercuspidals) there is a lot of well thought out, beautiful theory that goes into the 'proper' definition of L-functions but it takes a lot of setup. So, we bypass this just by giving the seemingly arbitrary 'answer'.

So, let us define the L-function of an irreducible admissible representation ρ of $\mathrm{GL}_2(\mathbb{Q}_p)$ as follows:

$$L(\rho, s) := \begin{cases} \dfrac{1}{(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})} & \text{if} \quad \rho = P(\chi_1, \chi_2),\ \alpha_i = \chi_i(p) \text{ if } \chi_i \text{ unramified } 0 \text{ otherwise} \\ \dfrac{1}{1-\alpha p^{-s}} & \text{if} \quad \rho = \mathrm{St}_{\mathrm{GL}_2} \otimes \chi \text{ and } \alpha = \chi(p) p^{\frac{-1}{2}} \text{ if } \chi |\cdot|_p^{\frac{1}{2}} \text{ unramified } 0 \text{ otherwise} \\ 1 & \text{if} \quad \rho \text{ supercuspidal} \end{cases}$$

One of the key things we'll see next time is how if π_f is the automorphic representation associated to f a Hecke eigencuspform, and $\pi_{f,p}$ denotes it's p-adic local factor (in the Flath decomposition) then $L_p(f, s) = L(\pi_{f,p}, s)$.